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ON THE ASYMPTOTIC INTEGRATION OF THE THREE-DIMENSIONAL NON-LINEAR EQUATIONS OF THIN ELASTIC SHELLS AND PLATES

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Abstract—Two-dimensional iterative procedures for the determination of the components of the stress tensor and of the displacement vector in thin anisotropic shells (plates) are derived from the three-dimensional (geometrically) non-linear equations of the elastic continuum theory by means of the method of asymptotic integration. The conditions, both for cases when the main system of equations of the iterative process is linear and for cases when the main system of the zeroth order approximation is non-linear, are given in terms of characteristic quantities, which characterise geometric and material properties of the shell (plate), and the intensity and the variability of the surface load. The attention is confined to the interior problem, the discussion of edge effects is omitted.

INTRODUCTION

THE problem considered in this paper is the derivation of approximate two-dimensional theories of thin plates and shells starting from the three-dimensional geometrically non-linear equations of elastic continuum theory. The method of derivation adopted here is an extension of the method of asymptotic integration used in a series of articles concerning the linear theory of shells and plates (see, e.g. [1–4]).

Recently, Rutten [5] has presented a different asymptotic approach in comparison with those of [1–4]. Employing the extended Chien's [6] method the two-dimensional interior shell equations are derived there first, and only then is the usual asymptotic technique adopted for an analysis of these equations. Various types of edge effects are also discussed in [5].

From the numerous literature concerned with other methods of reducing three-dimensional problems to lower-dimensional problems only papers by John [7, 8], Sensening [9], Koiter [10, 11] and Habip [12] are mentioned here (survey papers of recent developments have been published recently by Koiter [13] and Gol'denveizer [14]).

Especially Refs. [7–9] are of great interest. On using standard techniques from the theory of elliptic partial differential equations the derivation of approximate equations is accomplished there with certain estimates of errors made in the analysis.

The present paper begins with a restatement of the fundamental equations of the geometrically non-linear theory of elasticity for thin shells and plates of a homogeneous anisotropic material, having one plane of elastic symmetry which is parallel to the middle surface of the shell. These equations may be expressed in a number of alternative forms. For our purpose it is most convenient to use the non-linear equations in terms of a reference state as a basis for further investigation. The same form of the system of non-linear three-dimensional equations was used by Habip [12] for the derivation of two-dimensional non-linear field equations of elastic shell theory in terms of a reference state by means of the modified Hellinger–Reissner variational theorem. It may be of interest to compare the derivation and the resulting equations in [12] with those presented in this paper.

The system of equations given in Section 1 includes coefficients and unknown functions of different physical nature, and therefore, is not suitable for the order-of-magnitude analysis. To put these equations in the appropriate form, in Section 2 the non-dimensional components of stress and displacements are introduced and the equations are transformed into a new set of non-dimensional variables in which derivatives of the unknown functions are of the same order of magnitude as the functions themselves. Then, some further simple rearrangements are made, in consequence of which the non-dimensional coefficients in the transformed system are of unit magnitude at the maximum. In the non-dimensional form of the system of equations, the ratios of quantities characterising geometrical and material properties of the shell as well as qualitative and quantitative properties of an external load appear, and when considering these ratios as powers of a small parameter, we finally get the system of equations with the small parameter.

This system of equations is solved in Section 3 by the usual asymptotic procedure: the unknown functions are assumed to have the form of asymptotic expansions, these are then substituted into the equations and the coefficients of the equal powers of the small parameter are then equated. The structure of the obtained iterative process depends on mutual relations among the ratios of characteristic quantities. The limit is given when the state of stress and strain in the shell can be determined according to the linear iterative process.

The non-linear equations of the zeroth order approximation are given in Section 4. These equations are in complete agreement with those of the classical non-linear theory of plates and shells.

As the attention in this paper is confined to the so-called interior problem, we are omitting the discussion of the edge effects.

1. FUNDAMENTAL EQUATIONS

The system of fundamental (geometrically) non-linear three-dimensional equations of the elastic continuum theory, which will be analysed in this paper, consists of three equations of static equilibrium (without body forces) and six linear stress–strain relations, in which the components of the strain tensor are expressed in terms of components of the displacement vector. The unknown quantities are, therefore, the six independent components of the stress tensor and the three components of the displacement vector. As was pointed out in the introduction, it is most convenient to write the basic equations of a continuum theory in terms of a reference state. The stress state of a shell will then be defined according to [15] by the symmetric stress tensor s^{ij} measured per unit area of the undeformed body, and the components of the displacement vector are referred to the base vectors of the undeformed body.

Adapting the notation used by [3] the basic equations are

$$\mathbf{T}_{i,i} = 0, \qquad \gamma_{ij} = F_{ijkl}s^{kl}, \qquad (\ )_{,i} = \partial/\partial\vartheta^i \,(i = 1, 2, 3). \tag{1.1}$$

In equation (1.1), $\mathbf{T}_i\, \mathrm{d}\vartheta^j\, \mathrm{d}\vartheta^k\, (i \neq j \neq k)$ is the force vector acting on an element of area in the surface $\vartheta^i = \text{const}$, F_{ijkl} are the elastic coefficients of the body and γ_{ij} is the strain tensor. For $\mathbf{T}_i$ and γ_{ij} we have

$$\mathbf{T}_i = \sqrt{(g)}s^{ij}\mathbf{G}_j = \sqrt{(g)}t^{ij}\mathbf{g}_j, \qquad g = \det g_{ij},$$
$$\gamma_{ij} = \tfrac{1}{2}(\mathbf{g}_i \cdot \mathbf{V}_{,j} + \mathbf{g}_j \cdot \mathbf{V}_{,i} + \mathbf{V}_{,i} \cdot \mathbf{V}_{,j}), \tag{1.2}$$

where $\mathbf{V}$ is a displacement vector, $\mathbf{g}_i$ and g_{ij} are the base vectors and the metric tensor of the undeformed body respectively, and $\mathbf{G}_i$ are the base vectors of the deformed body. The fundamental equations (1.1) and (1.2) are written in the system of curvilinear convected coordinates ϑ^i (which deform continuously with the body). In the following, we identify the general convected coordinate system ϑ^i with a set of normal convected coordinates in which the points of a shell are defined by the position vector

$$\mathbf{R}(\vartheta^i) = \mathbf{r}(\vartheta^\alpha) + \vartheta^3\mathbf{a}_3(\vartheta^\alpha), \qquad -\frac{h}{2} \leq \vartheta^3 \leq \frac{h}{2}, \qquad (\alpha = 1, 2) \tag{1.3}$$

where $\mathbf{r}$ is the position vector of points of the middle surface of the shell, $\mathbf{a}_3$ is a unit normal to the middle surface and h is the constant thickness of the shell. The base vectors and the metric tensor associated with the coordinate system ϑ^α in the middle surface, the curvature tensor and the Gaussian curvature of the shell middle surface are denoted by $\mathbf{a}_\alpha$, $a_{\alpha\beta}$, $b_{\alpha\beta}$ and K respectively.
From (1.3) there follows

$$\mathbf{g}_\alpha = \mathbf{R}_{,\alpha} = \mu_\alpha^\lambda \mathbf{a}_\lambda, \qquad \mu_\alpha^\lambda = \delta_\alpha^\lambda - \vartheta^3 b_\alpha^\lambda,$$
$$g_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta = \mu_\alpha^\lambda \mu_\beta^\varkappa a_{\lambda\varkappa}, \qquad a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta = \mathbf{r}_{,\alpha} \cdot \mathbf{r}_{,\beta}, \tag{1.4}$$
$$\mu = \det \mu_\beta^\alpha = 1 - \vartheta^3 b_\lambda^\lambda + (\vartheta^3)^2 K, \qquad K = \det b_\beta^\alpha.$$

The displacement vector $\mathbf{V}$ can be expressed in alternative forms as

$$\mathbf{V} = v^i\mathbf{g}_i = v_i\mathbf{g}^i = u^\alpha\mathbf{a}_\alpha + w\mathbf{a}_3 = u_\alpha\mathbf{a}^\alpha + w\mathbf{a}_3 \tag{1.5}$$

and its derivatives with respect to ϑ^i are

$$\mathbf{V}_{,\alpha} = (\nabla_\alpha u_\lambda - b_{\alpha\lambda}w)\mathbf{a}^\lambda + (\nabla_\alpha w + b_\alpha^\lambda u_\lambda)\mathbf{a}_3,$$
$$\mathbf{V}_{,3} = u_{\alpha,3}\mathbf{a}^\alpha + w_{,3}\mathbf{a}_3, \tag{1.6}$$

where ∇_α denotes the covariant differentiation with respect to ϑ^α using the metric tensor of the undeformed body.

As we restrict our attention only to shells of homogeneous anisotropic material having one plane of elastic symmetry, which is perpendicular to the normal to the middle surface, the number of non-zero coefficients F_{ijkl} in (1.1) will be 13 (coefficients of the type $F_{\alpha 333}$, $F_{\alpha\beta\gamma 3}$ vanish). All non-zero elastic coefficients are shifted parallel to the middle surface (using μ_β^α as a shifter according to [16]) and the shifted coefficients are developed into Taylor series in the variable ϑ^3. In this way we obtain relations

$$F_{\alpha\beta\gamma\varkappa} = \mu_\alpha^\lambda \mu_\beta^\varrho \mu_\gamma^\omega \mu_\varkappa^\nu \sum_{l=0} (\vartheta^3)^l F_{\lambda\varrho\omega\nu}^{[l]}, \qquad F_{3333} = F_{3333}(\vartheta^\alpha),$$
$$F_{\alpha 3\beta 3} = \mu_\alpha^\lambda \mu_\beta^\varrho \sum_{l=0} (\vartheta^3)^l F_{\lambda 3\varrho 3}^{[l]}, \qquad F_{\alpha\beta 33} = \mu_\alpha^\lambda \mu_\beta^\varrho \sum_{l=0} (\vartheta^3)^l F_{\lambda\varrho 33}^{[l]}. \tag{1.7}$$

Temporarily using the notation

$$\mathbf{T}_i = (\bar{t}^{i\lambda}\mathbf{a}_\lambda + \bar{t}^{i3}\mathbf{a}_3)\sqrt{(a)}, \tag{1.8}$$

where from (1.2)

$$\bar{t}^{i\lambda} = \mu\mu^\lambda_\alpha t^{i\alpha}, \qquad \bar{t}^{i3} = \mu t^{i3}, \tag{1.9}$$

we can directly write down equilibrium equations by using the derivation from Ref. [3]

$$\nabla_\alpha \bar{t}^{\alpha\beta} - b^\beta_\alpha \bar{t}^{\alpha 3} + \bar{t}^{3\beta}_{,3} = 0, \qquad \nabla_\alpha \bar{t}^{\alpha 3} + b_{\alpha\beta}\bar{t}^{\alpha\beta} + \bar{t}^{33}_{,3} = 0. \tag{1.10}$$

If we substitute (1.9) in (1.10) using the expressions (see [15, 16])

$$t^{ij} = s^{ir}(\delta^j_r + v^j|_r), \tag{1.11}$$

and

$$\begin{aligned} v^\alpha|_\beta &= (\mu^{-1})^\alpha_\lambda(\nabla_\beta u^\lambda - b^\lambda_\beta w), \qquad v^\alpha|_3 = (\mu^{-1})^\alpha_\lambda u^\lambda_{,3}, \\ v^3|_\alpha &= \nabla_\alpha w + b_{\alpha\lambda}u^\lambda, \qquad v^3|_3 = w_{,3}, \qquad ((\mu^{-1})^\alpha_\beta \mu^\beta_\gamma = \delta^\alpha_\gamma) \end{aligned} \tag{1.12}$$

where the vertical line denotes the covariant derivative with respect to the coordinates ϑ^i in the undeformed body, and if we substitute (1.6) and (1.7) into the second equation in (1.1), then we can write the basic system of equations for the determination of the state of stress and strain in the shell in the form.

$$\begin{aligned} &\nabla_\alpha[\sigma^{\alpha\gamma}(\mu^\beta_\gamma + \nabla_\gamma u^\beta - b^\beta_\gamma w) + \sigma^{\alpha 3}u^\beta_{,3}] - b^\beta_\alpha[\sigma^{\alpha\gamma}(\nabla_\gamma w + b_{\gamma\lambda}u^\lambda) + \sigma^{\alpha 3}(1 + w_{,3})] \\ &\quad + [\sigma^{\gamma 3}(\mu^\beta_\gamma + \nabla_\gamma u^\beta - b^\beta_\gamma w) + \sigma^{33}u^\beta_{,3}]_{,3} = 0, \\ &\nabla_\alpha[\sigma^{\alpha\gamma}(\nabla_\gamma w + b_{\gamma\lambda}u^\lambda) + \sigma^{\alpha 3}(1 + w_{,3})] + b_{\alpha\beta}[\sigma^{\alpha\gamma}(\mu^\beta_\gamma + \nabla_\gamma u^\beta - b^\beta_\gamma w) + \sigma^{\alpha 3}u^\beta_{,3}] \\ &\quad + [\sigma^{3\gamma}(\nabla_\gamma w + b_{\gamma\lambda}u^\lambda) + \sigma^{33}(1 + w_{,3})]_{,3} = 0, \\ &\tfrac{1}{2}\mu[(\mu^\lambda_\alpha\nabla_\beta + \mu^\lambda_\beta\nabla_\alpha)u_\lambda - (\mu^\lambda_\alpha b_{\lambda\beta} + \mu^\lambda_\beta b_{\lambda\alpha})w + (\nabla_\alpha u_\lambda - b_{\alpha\lambda}w)(\nabla_\beta u^\lambda - b^\lambda_\beta w) + (\nabla_\alpha w + b^\lambda_\alpha u_\lambda) \\ &\quad \times (\nabla_\beta w + b^\nu_\beta u_\nu)] = \mu^\lambda_\alpha \mu^\varrho_\beta \mu^\omega_\gamma \mu^\nu_\varkappa \sum_{l=0} (\vartheta^3)^l F^{[l]}_{\lambda\varrho\omega\nu}\sigma^{\gamma\varkappa} + \mu^\lambda_\alpha\mu^\varrho_\beta \sum_{l=0} (\vartheta^3)^l F^{[l]}_{\lambda\varrho 33}\sigma^{33}, \end{aligned} \tag{1.13}$$

$$\begin{aligned} &\tfrac{1}{2}\mu[\mu^\lambda_\alpha u_{\lambda,3} + \nabla_\alpha w + b^\lambda_\alpha u_\lambda + (\nabla_\alpha u_\lambda - b_{\alpha\lambda}w)u^\lambda_{,3} + (\nabla_\alpha w + b^\lambda_\alpha u_\lambda)w_{,3}] \\ &\quad = 2\mu^\lambda_\alpha \mu^\varrho_\beta \sum_{l=0} (\vartheta^3)^l F^{[l]}_{\lambda 3\varrho 3}\sigma^{\beta 3}, \\ &\mu[w_{,3} + \tfrac{1}{2}((w_{,3})^2 + u_{\alpha,3}u^\alpha_{,3})] = F_{3333}\sigma^{33} + \mu^\lambda_\alpha\mu^\varrho_\beta \sum_{l=0} (\vartheta^3)^l F^{[l]}_{33\lambda\varrho}\sigma^{\alpha\beta}, \end{aligned}$$

where

$$\sigma^{ij} = \mu s^{ij} \qquad (\sigma^{ij} = \sigma^{ji}). \tag{1.14}$$

The solution of this system of equations should, furthermore, satisfy the following boundary conditions on the deformed faces of the shell

$$\mu\,{}_0\mathbf{t} = \sigma^{ij}\,{}_0 n_i \mathbf{G}_j = \mathbf{P}, \tag{1.15}$$

where ${}_0n_i$ are the components of the unit normal ${}_0\mathbf{n}$ to the faces of the shell in its undeformed state

$$ {}_0\mathbf{n} = {}_0n_i\mathbf{g}^i, \tag{1.16}$$

and the components of the external load vector $\mathbf{P}$ in (1.15), are measured per unit area of the middle surface of the undeformed body. As

$$ {}_0n_i = \pm\delta_i^3 \tag{1.17}$$

on the faces $\vartheta^3 = \pm\frac{1}{2}h$, the surface conditions become

$$\begin{aligned} [\sigma^{i3}]_{\vartheta^3=\frac{1}{2}h} &= P^i_+ = \mathscr{P}Q^i_+ ,\\ [\sigma^{i3}]_{\vartheta^3=-\frac{1}{2}h} &= P^i_- = \mathscr{P}Q^i_- , \end{aligned} \tag{1.18}$$

where P^i_+, P^i_- are components of the external load vector $\mathbf{P}$ acting on faces $\vartheta^3 = \frac{1}{2}h$, $\vartheta^3 = -\frac{1}{2}h$ respectively,

$$\begin{aligned} \mathbf{P} &= P^i_+\mathbf{G}_i & \vartheta^3 &= \tfrac{1}{2}h,\\ \mathbf{P} &= -P^i_-\mathbf{G}_i & \vartheta^3 &= -\tfrac{1}{2}h, \end{aligned} \tag{1.19}$$

and the quantity $\mathscr{P}$ in (1.18) is chosen in such a manner that

$$\max(Q^3_+ , Q^3_-) \approx 1. \tag{1.20}$$

Generally, the edge conditions should also be taken into account when choosing $\mathscr{P}$ but the former are left out from our consideration since only the interior shell problem is investigated here.

In section 3 some order-of-magnitude restrictions on the ratios Q^α/Q^3 will be given.

2. CHARACTERISTIC QUANTITIES. NON-DIMENSIONAL FORM OF THE FUNDAMENTAL EQUATIONS

The system of basic equations (1.13) with surface conditions (1.18) will be now treated by the method of asymptotic integration. At first, the system (1.13) is rewritten in non-dimensional form. On account of this arrangement the ratios of various quantities characterising the geometrical and material properties of a shell, and the quantitative and qualitative properties of the external load (which is assumed to be sufficiently smooth) appear explicitly in the transformed system of equations. Taking these ratios as powers of the small parameter we have to solve a system of equations with small parameter, which we treat by the usual asymptotic procedure.

We begin with an analysis of geometrical properties of the shell. In this analysis we use normal coordinates satisfying

$$a_{\alpha\beta} \approx 1. \tag{2.1}$$

The position of the point of the shell is determined by values of coordinates ϑ^i, which without loss of generality may be assumed to range over intervals

$$0 \le \vartheta^\alpha \le \beta^\alpha, \qquad -\frac{h}{2} \le \vartheta^3 \le \frac{h}{2}, \tag{2.2}$$

and further we consider only the case, when

$$\beta^1 \approx \beta^2. \tag{2.3}$$

Geometrical properties of the shell can be characterised by quantities h, β and R, where h is the constant thickness of the shell, β is the characteristic diameter of the region of the space which is occupied by the shell ($\beta \approx \beta^\alpha$) and R is the "generalised" characteristic radius of curvature of the shell middle surface satisfying relations (see also F. John [7])

$$b^\alpha_\beta \approx R^{-1}, \qquad \nabla_\alpha b^\beta_\gamma \approx R^{-2}, \qquad \nabla_\alpha \nabla_\beta b^\gamma_\varkappa \approx R^{-3}, \ldots \tag{2.4}$$

The material property of the considered shell is characterised by the quantity E from

$$F^{[l]}_{ijrs} = E^{-1} R^{-l} G^{[l]}_{ijrs} \qquad (l = 0, 1, 2, \ldots), \tag{2.5}$$

where E is defined by the requirement, that the following relations are satisfied

$$G^{[l]}_{ijrs} \approx 1, \qquad (G^{[0]}_{ijrs} = E[F_{ijrs}]_{\vartheta^3=0} = 0). \tag{2.6}$$

The quantitative characteristic of the external load is quantity $\mathscr{P}$ from (1.18). Finally, we introduce quantity L—the smallest "wave" length of the deformation and stress pattern on the middle surface, defined by (see Koiter [10])

$$\left|\frac{\mathrm{d}\mathscr{U}}{\mathrm{d}s}\right| \approx L^{-1}|\mathscr{U}|, \tag{2.7}$$

where $\mathscr{U}$ stands for arbitrary unknown quantity from (1.13) and ds is any arc element on the middle surface.

For our purposes we shall use (2.7) in the alternative form

$$|\nabla_\alpha \mathscr{U}| \approx L^{-1}|\mathscr{U}|. \tag{2.8}$$

Our problem is thus characterised by the "length" quantities h, β, R, L and by "force-per-unit-area" quantities E and $\mathscr{P}$. Except for the quantity L, all other characteristic quantities can be determined directly from their definition. Since for a thin shell we assume that the pattern of components of the stress resembles that of their prescribed values on the faces of the shell, we shall determine quantity L from

$$\nabla_\alpha[P] \approx L^{-1}[P], \tag{2.9}$$

where $[P]$ stands for an arbitrary component of the external load, and L is, therefore, a qualitative characteristic of the external load. Now we put

$$\frac{h}{2\beta} = \eta^{-q}, \qquad \frac{\beta}{L} = \eta^{p}, \qquad \frac{\beta}{R} = \eta^{-p^+}, \qquad \frac{E}{\mathscr{P}} = \eta^{\rho}, \tag{2.10}$$

where η is chosen in such a manner that q, p, p^+, ρ are integers (with $q > 0$). The quantity η^{-1} will be the small parameter in system (1.13) after its transformation into new co-ordinates, as will be shown later on, and we shall then treat this system for $\eta^{-1} \to 0$.

Proceeding further, we introduce non-dimensional forms for the components σ^{ij}, u_α, w

$$\sigma^{ij} = \mathscr{P}S^{ij}, \qquad u_\alpha = \beta U_\alpha, \qquad u^\alpha = \beta U^\alpha, \qquad w = \beta W \tag{2.11}$$

and we also perform a transformation of coordinates ϑ^i into a set of new non-dimensional variables x^i

$$x^\alpha = L^{-1}\vartheta^\alpha, \qquad x^3 = \frac{2}{h}\vartheta^3. \tag{2.12}$$

In coordinates x^α we have instead of (2.8) the relation

$$\nabla^*_\alpha \mathcal{U} \approx \mathcal{U} \tag{2.13}$$

where ∇^*_α is the symbol of covariant differentiation with respect to coordinates $x^\alpha(\nabla^*_\alpha = L\nabla_\alpha)$, and $\mathcal{U}$ stands for U_α, W, S^{ij}. Performing the above described transformations, taking (2.10) into consideration, and using the notation

$$\frac{h}{2R} = \eta^{-q^+}, \qquad q^+ = q+p^+, \tag{2.14}$$

we write (1.13) in the following form:

$$\begin{aligned}
&\eta^p\nabla^*_\alpha[S^{\alpha\gamma}(\delta^\beta_\gamma - \eta^{-q^+}x^3Rb^\beta_\gamma + \eta^p\nabla^*_\gamma U^\beta - \eta^{-p^+}Rb^\beta_\gamma W)\\
&\quad + \eta^q S^{\alpha 3}\partial_3 U^\beta] - \eta^{-(q^+ + p^+)}x^3R^2(\nabla_\alpha b^\beta_\gamma)S^{\alpha\gamma} - \eta^{-2p^+}R^2(\nabla_\alpha b^\beta_\gamma)S^{\alpha\gamma}W\\
&\quad - \eta^{-p^+}Rb^\beta_\alpha[S^{\alpha\gamma}(\eta^p\nabla^*_\gamma W + \eta^{-p^+}Rb_{\gamma\lambda}U^\lambda) + S^{\alpha 3}(1+\eta^q\partial_3 W)] + \eta^q\partial_3[S^{\gamma 3}(\delta^\beta_\gamma\\
&\quad - \eta^{-q^+}x^3Rb^\beta_\gamma + \eta^p\nabla^*_\gamma U^\beta - \eta^{-p^+}Rb^\beta_\gamma W) + \eta^q S^{33}\partial_3 U^\beta] = 0,\\
&\eta^p\nabla^*_\alpha[S^{\alpha\gamma}(\eta^p\nabla^*_\gamma W + \eta^{-p^+}Rb_{\gamma\lambda}U^\lambda) + S^{\alpha 3}(1+\eta^q\partial_3 W)] + \eta^{-2p^+}R^2(\nabla_\alpha b_{\gamma\lambda})S^{\alpha\gamma}U^\lambda\\
&\quad + \eta^{-p^+}Rb_{\alpha\beta}[S^{\alpha\gamma}(\delta^\beta_\gamma - \eta^{-q^+}x^3Rb^\beta_\gamma + \eta^p\nabla^*_\gamma U^\beta - \eta^{-p^+}Rb^\beta_\gamma W) + \eta^q S^{\alpha 3}\partial_3 U^\beta]\\
&\quad + \eta^q\partial_3[S^{3\gamma}(\eta^p\nabla^*_\gamma W + \eta^{-p^+}Rb_{\gamma\lambda}U^\lambda) + S^{33}(1+\eta^q\partial_3 W)] = 0,\\
&\tfrac{1}{2}\eta^p\sum_{j=0}^{2}(-1)^j(x^3)^j\eta^{-jq^+}R^j(I_{0j})\{\eta^p(\nabla^*_\alpha U_\beta + \nabla^*_\beta U_\alpha) - 2\eta^{-p^+}Rb_{\alpha\beta}W\\
&\quad - \eta^{-q^+}x^3[\eta^pR(b^\lambda_\alpha\nabla^*_\beta \pm b^\lambda_\beta\nabla^*_\alpha)U_\lambda - \eta^{-p^+}R^2(b^\lambda_\alpha b_{\lambda\beta} + b^\lambda_\beta b_{\lambda\alpha})W] + (\eta^p\nabla^*_\alpha U_\lambda\\
&\quad - \eta^{-p^+}Rb_{\alpha\lambda}W)(\eta^p\nabla^*_\beta U^\lambda - \eta^{-p^+}Rb^\lambda_\beta W) + (\eta^p\nabla^*_\alpha W + \eta^{-p^+}Rb^\lambda_\alpha U_\lambda)(\eta^p\nabla^*_\beta W + \eta^{-p^+}Rb^\nu_\beta U_\nu)\}\\
&\quad = \sum_{j=0}^{4}\sum_{l=0}(-1)^j(x^3)^{j+l}\eta^{-(j+l)q^+}R^j(I_{4j})^{\lambda\varrho\omega\nu}_{\alpha\beta\gamma\varkappa}G^{[l]}_{\lambda\varrho\omega\nu}S^{\gamma\varkappa}\\
&\qquad + \sum_{j=0}^{2}\sum_{l=0}(-1)^j(x^3)^{j+l}\eta^{-(j+l)q^+}R^j(I_{2j})^{\lambda\varrho}_{\alpha\beta}G^{[l]}_{\lambda\varrho 33}S^{33},
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
&\tfrac{1}{2}\eta^p\sum_{j=0}^{2}(-1)^j(x^3)^j\eta^{-jq^+}R^j(I_{0j})\{\eta^q\partial_3 U_\alpha + \eta^p\nabla^*_\alpha W + \eta^{-p^+}Rb^\lambda_\alpha(U_\lambda - x^3\partial_3 U_\lambda)\\
&\quad + \eta^q\partial_3 U^\lambda(\eta^p\nabla^*_\alpha U_\lambda - \eta^{-p^+}Rb_{\alpha\lambda}W) + \eta^q\partial_3 W(\eta^p\nabla^*_\alpha W + \eta^{-p^+}Rb^\lambda_\alpha U_\lambda)\}\\
&\quad = 2\sum_{j=0}^{2}\sum_{l=0}(-1)^j(x^3)^{j+l}\eta^{-(j+l)q^+}R^j(I_{2j})^{\lambda\varrho}_{\alpha\beta}G^{[l]}_{\lambda 3\varrho 3}S^{\beta 3},\\
&\eta^p\sum_{j=0}^{2}(-1)^j(x^3)^j\eta^{-jq^+}R^j(I_{0j})\{\eta^q\partial_3 W + \tfrac{1}{2}\eta^{2q}[(\partial_3 W)^2 + (\partial_3 U_\alpha)\partial_3 U^\alpha]\}\\
&\quad = \sum_{j=0}^{2}\sum_{l=0}(-1)^j(x^3)^{j+l}\eta^{-(j+l)q^+}R^j(I_{2j})^{\lambda\varrho}_{\alpha\beta}G^{[l]}_{33\lambda\varrho}S^{\alpha\beta} + G_{3333}S^{33},
\end{aligned}$$

where

$$\partial_3 = \partial/\partial x^3, \quad (I_{00}) = 1, \quad (I_{01}) = b_\lambda^\lambda, \quad (I_{02}) = K,$$
$$(I_{ij})^{\alpha\ldots\beta\ldots}_{\lambda\ldots\varrho\ldots} = \sum \underbrace{\delta^\alpha_\lambda \delta^\cdot_\cdot \delta^\cdot_\cdot}_{i-j} \underbrace{b^\beta_\varrho b^\cdot_\cdot b^\cdot_\cdot}_{j}. \tag{2.16}$$

In the sum from (2.16), the summed up terms are of the same pattern for the δ and b factors, differing from each other only by the combination of couples of subscripts and superscripts. Hence, the number of the summed terms is equal to the number of combination of i things $(i-j)$ at a time.

Let it be illustrated by the following example

$$(I_{42})^{\lambda\varrho\omega\nu}_{\alpha\beta\gamma\varkappa} = \delta^\lambda_\alpha \delta^\varrho_\beta b^\omega_\gamma b^\nu_\varkappa + \delta^\lambda_\alpha \delta^\omega_\gamma b^\varrho_\beta b^\nu_\varkappa + \delta^\lambda_\alpha \delta^\nu_\varkappa b^\varrho_\beta b^\omega_\gamma$$
$$+ \delta^\varrho_\beta \delta^\omega_\gamma b^\lambda_\alpha b^\nu_\varkappa + \delta^\varrho_\beta \delta^\nu_\varkappa b^\lambda_\alpha b^\omega_\gamma + \delta^\omega_\gamma \delta^\nu_\varkappa b^\lambda_\alpha b^\varrho_\beta.$$

Furthermore, we must keep in mind that when arranging the notation of (2.15) into the possibly most compact form we had to prescribe the following property to the operation ∇^*_α

$$\nabla^*_\gamma b_{\alpha\beta} \equiv 0. \tag{2.17}$$

When applying the foregoing asymptotic procedure to the surface conditions (1.18) we obtain

$$[S^{i3}]_{x^3=1} = Q^i_+,$$
$$[S^{i3}]_{x^3=-1} = Q^i_-. \tag{2.18}$$

3. METHOD OF SOLUTION AND RESULTING FORM OF ITERATIVE PROCESSES

The solution of the system of equations (2.15) subject to the surface conditions (2.18) is assumed to have the form

$$S^{ij} = \eta^{r_{ij}} \sum_{s=0} \eta^{-s} S^{ij}_{[s]}, \qquad W = \eta^{r_3} \sum_{s=0} \eta^{-s} W^{[s]},$$
$$U_\alpha = \eta^{r_\alpha} \sum_{s=0} \eta^{-s} U^{[s]}_\alpha, \qquad s = t/a, \qquad t = 0, 1, 2, \ldots, \tag{3.1}$$

where a is a constant given in (3.7).

The components of the external load are also expanded into asymptotic series

$$Q^i_+ = \eta^{r_{i3}} \sum_{s=0} \eta^{-s} Q^i_{[s]+}, \qquad Q^i_- = \eta^{r_{i3}} \sum_{s=0} \eta^{-s} Q^i_{[s]-}. \tag{3.2}$$

In what follows only the surface load satisfying

$$Q^\alpha \approx \eta^{\varkappa_\alpha} Q^3, \qquad \varkappa_\alpha \leq r_{\alpha 3} - r_{33}, \tag{3.3}$$

will be considered.

Notice that from (1.20), (3.2) and (3.3) we have

$$Q^\alpha_{[s]+} = Q^\alpha_{[s]-} \equiv 0 \quad \text{if} \quad s < r_{\alpha 3} - \varkappa_\alpha \qquad (r_{33} = 0). \tag{3.4}$$

The iterative procedure for successive determination of the coefficients of the asymptotic expansions (3.1) can now be obtained by the usual procedure of the asymptotic methods:

expressions (3.1) and (3.2) are substituted into (2.15) and (2.18) and the coefficients of powers of the small parameter η^{-1} are then equated in the system of equations obtained in this way.

In order to obtain a non-contradictory iterative process the set of exponents r from (3.1) must be adequately chosen in dependence on mutual relations existing among exponents q, p, p^+ and the exponent ρ must also satisfy an additional restricting inequality.

We have to consider separately three different cases, defined by the respective inequalities

$$\begin{aligned} &(A):\ p^+ < q-2p, \quad \rho \geq q^+ + 2p^+ + 2p \text{ (and simultaneously } \rho > q^+); \\ &(B):\ p^+ \geq q-2p, \quad \rho \geq 4(q-p); \\ &(C):\ q-p \leq \rho < \min[q^+ + 2p^+ + 2p, 4(q-p)] \end{aligned} \tag{3.5}$$

The corresponding non-contradictory sets of exponents r are

$$\begin{aligned} &(A):\ r_{33} = 0, \quad r_{\alpha 3} = p+p^+, \quad r_{\alpha\beta} = q^+, \quad r_\alpha = q^+ - p - \rho, \quad r_3 = q^+ + p^+ - \rho; \\ &(B):\ r_{33} = 0, \quad r_{\alpha 3} = q-p, \quad r_{\alpha\beta} = 2(q-p), \quad r_\alpha = 2q-3p-\rho, \quad r_3 = 3q-4p-\rho; \\ &(C):\ r_{33} = 0, \quad r_{\alpha 3} = \tfrac{1}{3}(p-q+\rho), \quad r_{\alpha\beta} = \tfrac{1}{3}(2q-2p+\rho), \quad r_\alpha = \tfrac{1}{3}(2q-5p-2\rho), \\ &\qquad r_3 = \tfrac{1}{3}(q-4p-\rho); \end{aligned} \tag{3.6}$$

and the constant a from (3.1) takes the respective values

$$\begin{aligned} a &= 1 \quad \text{in cases (A), (B);} \\ a &= 3 \quad \text{in case (C).} \end{aligned} \tag{3.7}$$

On using (3.6) and (3.7) and performing the above procedure of asymptotic integration we obtain the iterative process for the determination of the unknown quantities [of coefficients of (3.1)]. The iterative process still has a three-dimensional form, it includes derivatives of unknown quantities with respect to x^3 and powers of x^3 itself as coefficients.

The elimination of the independent variable x^3 may be performed by writing

$$S^{ij}_{[s]} = \sum_{n=0}^{\lambda_{ij}} (x^3)^n S^{ij}_{[s,n]}, \qquad U^{[s]}_\alpha = \sum_{n=0}^{\lambda_\alpha} (x^3)^n U^{[s,n]}_\alpha, \qquad W^{[s]} = \sum_{n=0}^{\lambda_3} (x^3)^n W^{[s,n]} \tag{3.8}$$

substituting (3.8) into equations of iterative process and by equating the coefficients of powers of x^3 (which procedure is equivalent to the integration with respect to x^3). The detailed analysis of the system (2.15) shows that the presented procedure of the derivation of a two-dimensional iterative process can be successfully applied when the exponent ρ satisfies the second inequalities from (3.5). In that way we obtain only one physically admissible two-dimensional iterative process since for $\rho \geq 2q^+$ in case (A) and $\rho > 4(q-p)$ in case (B) the three-dimensional form of the iterative process is linear in x^3 and for other possible cases from (3.5) the non-linear in x^3 approximation to the last equation from (2.15)

$$\begin{aligned} &(\partial_3 W^{[s]})^2 = 0 \quad \text{if } s < s^*, && s^* = 2q^+ - \rho && \text{(A)}, \\ &\partial_3 W^{[s^*]} + \tfrac{1}{2}(\partial_3 W^{[s^*]})^2 = 0, && s^* = 0 && \text{(B)}, \\ & && s^* = \tfrac{1}{3}(4q-4p-\rho) && \text{(C)}, \end{aligned} \tag{3.9}$$

has only one physically meaningful solution

$$W^{[s]} = W^{[s]}(x^\alpha), \qquad s \le s^*. \tag{3.10}$$

At this point, it is convenient to transform the resulting equations of the iterative process to the coordinates ϑ^α by performing inverse transformation to (2.12.1). Further, we turn back to the dimensional form of the components of stress and displacement σ^{ij}, u_α, w, and instead of (3.1), (3.2) we introduce the following expansions for the unknown components σ^{ij}, u_α, w and for the components P^i of the external load

$$\begin{gathered}
\sigma^{ij} = \sum_{s=0} \sum_{n=0}^{\lambda_{ij}} (x^3)^n \sigma^{ij}_{[s,n]}, \qquad u_\alpha = \sum_{s=0} \sum_{n=0}^{\lambda_\alpha} (x^3)^n u_\alpha^{[s,n]}, \\
w = \sum_{s=0} \sum_{n=0}^{\lambda_3} (x^3)^n w^{[s,n]}, \qquad P^i = \sum_{s=0} P^i_{[s]}; \\
(\sigma^{ij}_{[s,n]} = \mathscr{P}\eta^{r_{ij}-s} S^{ij}_{[s,n]}, \qquad u_\alpha^{[s,n]} = \beta\eta^{r_\alpha - s} U_\alpha^{[s,n]}, \qquad w^{[s,n]} = \beta\eta^{r_3 - s} W^{[s,n]}, \\
P^i_{[s]} = \mathscr{P}\eta^{r_{i3}-s} Q^i_{[s]})
\end{gathered} \tag{3.11}$$

In consequence of these rearrangements, the characteristic quantities β, R, L, $\mathscr{P}$, E, the choice of which is arbitrary to some extent, are eliminated from the resulting form of the iterative process used for the determination of the coefficients of expansions (3.11).

After introducing the notation

$$\begin{aligned}
\sum_{i+k}^{a} \sum_{l+m}^{n} \varkappa[xA \,.\, yB]_t &= \sum_{i+k=a_t} \sum_{l+m=n} \varkappa x A_{[i,l]} \cdot y B_{[k,m]}, \\
\sum_{i+k}^{a} \sum_{l+m}^{n} \varkappa[(l+1)xA \,.\, yB]_t &= \sum_{i+k=a_t} \sum_{l+m=n} \varkappa(l+1) x A_{[i,l+1]} \cdot y B_{[k,m]}, \\
\sum_{i+k}^{a} \sum_{l+m}^{n} \varkappa[(m+1)xA \,.\, yB]_t &= \sum_{i+k=a_t} \sum_{l+m=n} \varkappa(m+1) x A_{[i,l]} \cdot y B_{[k,m+1]}, \\
\sum_{i+k}^{a} \sum_{l+m}^{n} \varkappa[(l+1)(m+1)xA \,.\, yB]_t &= \sum_{i+k=a_t} \sum_{l+m=n} \varkappa(l+1)(m+1) x A_{[i,l+1]} \cdot y B_{[k,m+1]}, \\
\sum_{i+k}^{a} \sum_{l+m}^{n} \varkappa[(m+1)(m+2)xA \,.\, yB]_t &= \sum_{i+k=a_t} \sum_{l+m=n} \varkappa(m+1)(m+2) x A_{[i,l]} \cdot y B_{[k,m+2]},
\end{aligned} \tag{3.12}$$

(where $\varkappa$ is an arbitrary coefficient, x and y represent operators ∇_α or arbitrary coefficients, A and B represent any quantities from among σ^{ij}, u_α, w; i, k, l, m, t take values $0, 1, 2, \ldots$, and in the brackets the usual rules for arithmetic operations and for differentiation are preserved) we are in position to present a compact form of the relations for successive determination of the coefficients of the expansions (3.11)

$$1.\ \sigma^{\beta 3}_{[s,n]} = -\frac{h}{2n}\Bigg\{\nabla_\alpha \sigma^{\alpha\beta}_{[a_1,n-1]} - \frac{h}{2}[b^\beta_\gamma \nabla_\alpha \sigma^{\alpha\gamma}_{[a_2,n-2]} + (\nabla_\alpha b^\beta_\gamma)\sigma^{\alpha\gamma}_{[a_3,n-2]}]$$

$$-(n+1)b^\beta_\alpha \sigma^{\alpha 3}_{[a_4,n-1]} + \sum_{i+k}^{a}\sum_{l+m}^{n-1}\Bigg\{[\nabla_\alpha(\sigma^{\alpha\gamma}\,.\,\nabla_\gamma u^\beta)]_5 - b^\beta_\gamma[\nabla_\alpha(\sigma^{\alpha\gamma}\,.\,w)]_6$$

$$+\frac{2}{h}[(m+1)\nabla_\alpha(\sigma^{\alpha 3}\,.\,u^\beta)]_7 - (\nabla_\alpha b^\beta_\gamma)[\sigma^{\alpha\gamma}\,.\,w]_8 - b^\beta_\alpha[\sigma^{\alpha\gamma}\,.\,\nabla_\gamma w]_9 - b^\beta_\alpha b_{\gamma\lambda}[\sigma^{\alpha\gamma}\,.\,u^\lambda]_{10}$$

$$-\frac{2}{h}b^\beta_\alpha[(l+1)\sigma^{\alpha 3}\,.\,w + 2(m+1)\sigma^{\alpha 3}\,.\,w]_{11} + \frac{2}{h}[(l+1)\sigma^{\alpha 3}\,.\,\nabla_\alpha u^\beta$$

$$+(m+1)\sigma^{\alpha 3}\,.\,\nabla_\alpha u^\beta]_{12} + \left(\frac{2}{h}\right)^2[(l+1)(m+1)\sigma^{33}\,.\,u^\beta + (m+1)(m+2)\sigma^{33}\,.\,u^\beta]_{13}\Bigg\}\Bigg\}\ ;$$

$$(n = 1, 2, \ldots, \lambda_{\alpha 3}) \qquad (3.13)$$

$a_1 = s+p-q+r_{\alpha\beta}-r_{\alpha 3}$, $\quad a_8 = s-q^+-p^++r_{\alpha\beta}+r_3-r_{\alpha 3}$,

$a_2 = a_1-q^+$, $\quad a_9 = a_8+p+p^+$,

$a_3 = a_2-p-p^+$, $\quad a_{10} = s-q^+-p^++r_{\alpha\beta}+r_\alpha-r_{\alpha 3}$,

$a_4 = s-q^+$, $\quad a_{11} = s-p^++r_3$,

$a_5 = s+2p-q+r_{\alpha\beta}+r_\alpha-r_{\alpha 3}$, $\quad a_{12} = s+p+r_\alpha$,

$a_6 = s+p-q^++r_{\alpha\beta}+r_3-r_{\alpha 3}$, $\quad a_{13} = s+q+r_{33}+r_\alpha-r_{\alpha 3}$;

$a_7 = s+p+r_\alpha$,

$$2.\ \sigma^{33}_{[s,n]} = -\frac{h}{2n}\Bigg\{\nabla_\alpha \sigma^{\alpha 3}_{[b_1,n-1]} + b_{\alpha\beta}\sigma^{\alpha\beta}_{[b_2,n-1]} - \frac{h}{2}b_{\alpha\beta}b^\beta_\gamma \sigma^{\alpha\gamma}_{[b_3,n-2]}$$

$$+\sum_{i+k}^{b}\sum_{l+m}^{n-1}\Bigg\{[\nabla_\alpha(\sigma^{\alpha\gamma}\,.\,\nabla_\gamma w)]_4 + b_{\alpha\beta}[(\nabla_\gamma\sigma^{\alpha\gamma})\,.\,u^\beta + 2\sigma^{\alpha\gamma}\,.\,\nabla_\gamma u^\beta]_5$$

$$+(\nabla_\alpha b_{\gamma\lambda})[\sigma^{\alpha\gamma}\,.\,u^\lambda]_6 + \frac{2}{h}[(l+1)\sigma^{\gamma 3}\,.\,\nabla_\gamma w + (m+1)(\nabla_\gamma \sigma^{\gamma 3})\,.\,w$$

$$+2(m+1)\sigma^{\gamma 3}\,.\,\nabla_\gamma w]_7 - b_{\alpha\beta}b^\beta_\gamma[\sigma^{\alpha\gamma}\,.\,w]_8 + \frac{2}{h}b_{\alpha\beta}[(l+1)\sigma^{\alpha 3}\,.\,u^\beta + 2(m+1)\sigma^{\alpha 3}\,.\,u^\beta]_9$$

$$+\left(\frac{2}{h}\right)^2[(l+1)(m+1)\sigma^{33}\,.\,w + (m+1)(m+2)\sigma^{33}\,.\,w]_{10}\Bigg\}\Bigg\}, \qquad (n = 1, 2, \ldots, \lambda_{33});$$

$b_1 = s+p-q+r_{\alpha 3}-r_{33}$, $\quad b_6 = b_5-p-p^+$,

$b_2 = s-q^++r_{\alpha\beta}-r_{33}$, $\quad b_7 = s+p+r_{\alpha 3}+r_3-r_{33}$,

$b_3 = b_2-q^+$, $\quad b_8 = s-q^+-p^++r_{\alpha\beta}+r_3-r_{33}$,

$b_4 = s+2p-q+r_{\alpha\beta}+r_3-r_{33}$, $\quad b_9 = s-p^++r_{\alpha 3}+r_\alpha-r_{33}$,

$b_5 = s+p-q^++r_{\alpha\beta}+r_\lambda-r_{33}$, $\quad b_{10} = s+q+r_3$;

3.
$$\sigma^{\alpha\beta}_{[s,n]} = H^{\alpha\beta\gamma\varkappa}R^{[s,n]}_{\gamma\varkappa} \qquad (n = 0, 1, \ldots, \lambda_{\alpha\beta}),$$
$$H^{\alpha\beta\gamma\varkappa}F^{[0]}_{\gamma\varkappa\lambda\mu} = \tfrac{1}{2}\,(\delta^\alpha_\lambda\delta^\beta_\mu + \delta^\alpha_\mu\delta^\beta_\lambda),$$
$$H^{\alpha\beta\gamma\varkappa} = H^{\beta\alpha\gamma\varkappa} = H^{\alpha\beta\varkappa\gamma} = H^{\gamma\varkappa\alpha\beta},$$
$$R^{[s,n]}_{\alpha\beta} = \frac{1}{2}\sum_{j=0}^{2}(-1)^j\left(\frac{h}{2}\right)^j(I_{0j})\Bigg[\nabla_\alpha u^{[c_1,n-j]}_\beta + \nabla_\beta u^{[c_1,n-j]}_\alpha - 2b_{\alpha\beta}w^{[c_2,n-j]}$$
$$-\frac{h}{2}(b^\lambda_\alpha\nabla_\beta + b^\lambda_\beta\nabla_\alpha)u^{[c_3,n-j-1]}_\lambda + \frac{h}{2}(b^\lambda_\alpha b_{\lambda\beta} + b^\lambda_\beta b_{\lambda\alpha})w^{[c_4,n-j-1]}\Bigg]$$
$$-\sum_{i=0}\sum_{j=0}^{4}(-1)^j\{\chi_{ij}\}\left(\frac{h}{2}\right)^{i+j}(I_{4j})^{\lambda\varrho\omega\nu}_{\alpha\beta\gamma\varkappa}F^{[i]}_{\lambda\varrho\omega\nu}\sigma^{\gamma\varkappa}_{[c_5,n-i-j]}$$
$$-\sum_{i=0}\sum_{j=0}^{2}(-1)^j\left(\frac{h}{2}\right)^{i+j}(I_{2j})^{\lambda\varrho}_{\alpha\beta}F^{[i]}_{\lambda\varrho33}\sigma^{33}_{[c_6,n-i-j]}$$
$$+\frac{1}{2}\sum_{j=0}^{2}(-1)^j\left(\frac{h}{2}\right)^j(I_{0j})\Bigg\{\sum_{i+k}^{c}\sum_{l+m}^{n-j}\{[\nabla_\alpha w\,.\,\nabla_\beta w]_7 + [\nabla_\alpha u_\lambda\,.\,\nabla_\beta u^\lambda]_8$$
$$-[(b_{\lambda\alpha}\nabla_\beta + b_{\lambda\beta}\nabla_\alpha)u^\lambda\,.\,w - u_\lambda\,.\,(b^\lambda_\alpha\nabla_\beta + b^\lambda_\beta\nabla_\alpha)w]_9$$
$$+b_{\lambda\alpha}b^\lambda_\beta[w\,.\,w]_{10} + b^\lambda_\alpha b^\nu_\beta[u_\lambda\,.\,u_\nu]_{11}\}\Bigg\},$$
where $\{\chi_{00}\} = 0$, $\{\chi_{ij}\} = 1$ if simultaneously $i \neq 0$, $j \neq 0$,

$$c_1 = s+p+\rho-jq^+ + r_\alpha - r_{\alpha\beta}, \qquad c_7 = s+2p+\rho-jq^+ + 2r_3 - r_{\alpha\beta},$$
$$c_2 = s-p^+ +\rho-jq^+ + r_3 - r_{\alpha\beta}, \qquad c_8 = s+2p+\rho-jq^+ + 2r_\alpha - r_{\alpha\beta},$$
$$c_3 = c_1 - q^+, \qquad c_9 = s+p-p^+ +\rho-jq^+ + r_3 + r_\alpha - r_{\alpha\beta},$$
$$c_4 = c_2 - q^+, \qquad c_{10} = s-2p^+ +\rho-jq^+ + 2r_3 - r_{\alpha\beta},$$
$$c_5 = s-(i+j)q^+, \qquad c_{11} = s-2p^+ +\rho-jq^+ + 2r_\alpha - r_{\alpha\beta};$$
$$c_6 = s-(i+j)q^+ + r_{33} - r_{\alpha\beta},$$

4.
$$u^{[s,n]}_\alpha = -\frac{1}{n}\Bigg\{\sum_{j=0}^{2}(-1)^j\left(\frac{h}{2}\right)^j(I_{0j})\Bigg[\{\chi_j\}(n-j)u^{[d_1,n-j]}_\alpha + \frac{h}{2}\nabla_\alpha w^{[d_2,n-j-1]}$$
$$-\frac{h}{2}(n-j-2)b^\lambda_\alpha u^{[d_3,n-j-1]}_\alpha\Bigg] - 4\sum_{i=0}\sum_{j=0}^{2}(-1)^j\left(\frac{h}{2}\right)^{i+j+1}(I_{2j})^{\lambda\varrho}_{\alpha\beta}F^{[i]}_{\lambda3\varrho3}\sigma^{\beta3}_{[d_4,n-i-j-1]}$$
$$+\sum_{j=0}^{2}(-1)^j\left(\frac{h}{2}\right)^{j+1}(I_{0j})\Bigg\{\sum_{i+k}^{d}\sum_{l+m}^{n-j-1}\{[(l+1)u^\lambda\,.\,\nabla_\alpha u_\lambda]_5 - b_{\alpha\lambda}[(l+1)u^\lambda\,.\,w$$
$$-(m+1)u^\lambda\,.\,w]_6 + [(l+1)w\,.\,\nabla_\alpha w]_7\}\Bigg\}\Bigg\} \qquad (n = 1, 2, \ldots, \lambda_\alpha),$$
$$d_1 = s-jq^+, \qquad d_5 = s+p-jq^+ + r_\alpha,$$
$$d_2 = s-q+p-jq^+ + r_3 - r_\alpha, \qquad d_6 = s-p^+ -jq^+ + r_3,$$
$$d_3 = d_1 - q^+, \qquad d_7 = s+p-jq^+ + 2r_3 - r_\alpha;$$
$$d_4 = s-q-\rho-(i+j)q^+ + r_{\alpha3} - r_\alpha,$$

$$5.\ w^{[s,n]} = \frac{1}{n}\Bigg\{\sum_{i=0}^{2}\sum_{j=0}^{2}(-1)^j\left(\frac{h}{2}\right)^{i+j+1}(I_{2j})^{\lambda\varrho}_{\alpha\beta}F^{[i]}_{33\lambda\varrho}\sigma^{\alpha\beta}_{[e_1,n-i-j-1]}$$

$$+\frac{h}{2}F_{3333}\sigma^{33}_{[e_2,n-1]}-\sum_{j=0}^{2}(-1)^j\left(\frac{h}{2}\right)^j(I_{0j})\Bigg\{\{\chi_j\}(n-j)w^{[e_3,n-j]}$$

$$+\frac{1}{h}\sum_{i+k}^{e}\sum_{l+m}^{n-j-1}\{[(l+1)(m+1)w\,.\,w]_4+[(l+1)(m+1)u_\alpha\,.\,u^\alpha]_5\}\Bigg\}\Bigg\}$$

$(n = 1, 2, \ldots, \lambda_3)$,

$$e_1 = s-q-(i+j)q^+-\rho+r_{\alpha\beta}-r_3, \qquad e_4 = s+q-jq^++r_3,$$

$$e_2 = s-q-\rho+r_{33}-r_3, \qquad e_5 = s+q-jq^++2r_\alpha-r_3;$$

$$e_3 = s-jq^+,$$

It is to be noted that in (3.13)

$$\sigma^{ij}_{[s,n]} = w^{[s,n]} = u^{[s,n]}_\alpha \equiv 0 \quad \text{if} \quad s<0;\ n<0 \text{ or } n>\lambda, \tag{3.14}$$

and that the symbol $\{\chi_j\}$ in relations 4,5 of (3.13) has the following meaning

$$\{\chi_0\} = 0, \qquad \{\chi_j\} = 1 \quad \text{if} \quad j \neq 0. \tag{3.15}$$

Further, the exponent λ_3 in relation 5 of (3.13) should be taken as zero if $s < s^*$, s^* being given in (3.9). From the examination of (3.13) it is obvious that, provided the quantities $u^{[s,0]}_\alpha$, $w^{[s,0]}$, $\sigma^{i3}_{[s,0]}$ are known, all the other coefficients of expansions (3.11) with index s can be determined directly from (3.13) using only arithmetic operations and differentiation. The basic unknown quantities $u^{[s,0]}_\alpha$, $w^{[s,0]}$, $\sigma^{i3}_{[s,0]}$ can be determined from six surface conditions (2.18), which after performing the asymptotic procedure acquire the form

$$\left[\sum_{n=0}^{\lambda_{i3}}(x^3)^n\sigma^{i3}_{[s,n]}\right]_{x^3=1} = P^i_{[s]+},$$
$$\left[\sum_{n=0}^{\lambda_{i3}}(x^3)^n\sigma^{i3}_{[s,n]}\right]_{x^3=-1} = P^i_{[s]-}. \tag{3.16}$$

Taking the differences of these expressions, we obtain three equations from which quantities $\sigma^{i3}_{[s,0]}$ have explicitly disappeared

$$\sum_{n=0}^{\langle\lambda_{i3}/2\rangle}\sigma^{i3}_{[s,2n+1]} = \tfrac{1}{2}(P^i_{[s]+}-P^i_{[s]-}), \tag{3.17}$$

where $\langle\lambda\rangle$ represents the integer part of number λ. When considering cases (A) and (C) from (3.5) we substitute (3.13) into (3.17) and obtain three equations with three unknown functions $u^{[s,0]}_\alpha$, $w^{[s,0]}$. Case (A) includes shells loaded by a moderately varying surface load, as it can be established from (2.10) and (3.5), the variability of the surface load being characterized by quantity L from (2.9). Case (C) comprises flat and very shallow membranes (precisely speaking, it includes plates and (very shallow) shells the stresses and displacements of which should be determined in the lowest-order approximation from the equations of non-linear theory of flat membranes).

In case (B) the simple rearrangement of (3.17) has to be made to remove $\sigma^{33}_{[s,1]}$, since this term is dependent on the basic unknown quantities $\sigma^{\alpha 3}_{[s,0]}$ [see equation (3.13.2)]. To do this (3.13.2) is written in the following form

$$\sigma^{33}_{[s,n]} = -\frac{h}{2n}(\nabla_\alpha \sigma^{\alpha 3}_{[s,n-1]} + R^{33}_{[s,n]}). \tag{3.18}$$

The expression for $R^{33}_{[s,n]}$ can be easily found by comparison of (3.18) and (3.13.2). Making use of (3.18) and of the surface condition (3.16) with $i = 1, 2$, we obtain the following system of equations

$$\begin{aligned} \sum_{n=0}^{\langle \lambda_{\alpha 3}/2 \rangle} \sigma^{\alpha 3}_{[s,2n+1]} &= \tfrac{1}{2}(P^{\alpha}_{[s]+} - P^{\alpha}_{[s]-}), \\ \sum_{n=0}^{\langle \lambda_{33}/2 \rangle} n\sigma^{33}_{[s,2n+1]} &= -\frac{1}{4}\left[P^{3}_{[s]+} - P^{3}_{[s]-} + \frac{h}{2}\nabla_\alpha (P^{\alpha}_{[s]-} + P^{\alpha}_{[s]+})\right] - \frac{h}{4}\sum_{n=0}^{\langle \lambda_{33}/2 \rangle} R^{33}_{[s,2n+1]}. \end{aligned} \tag{3.19}$$

The expressions for $\sigma^{ij}_{[s,n]}$ from (3.13) are now substituted into (3.19) and system (3.19) assumes the resulting form with $u^{[s,0]}_{\alpha}$ and $w^{[s,0]}$ as the unknown functions that are to be determined.

Case (B) includes plates ($R = \infty$ and then $p^+ > c$, where c is an arbitrary constant), shallow shells [owing to relatively great value of R because the "shallowness" of the shell is indicated by exponent p^+ from (2.10)] and shells loaded by quickly varying surface load (on account of a relatively small value of L).

The resulting system of equations (3.17) or (3.19) will be called the main system of iterative process. This is linear provided

$$\begin{aligned} &\rho > q^+ + 2p^+ + 2p, \qquad &\text{(A)} \\ &\rho > 4(q-p). \qquad &\text{(B)} \end{aligned} \tag{3.20}$$

The non-linear main system of equations is derived for $u^{[0,0]}_{\alpha}$, $w^{[0,0]}$, if

$$\begin{aligned} &\rho = q^+ + 2p^+ + 2p \qquad (p+p^+ > 0), \qquad &\text{(A)} \\ &\rho = 4(q-p), \qquad &\text{(B)} \end{aligned} \tag{3.21}$$

and also when (C) is the case in question.

If the exponent ρ does not satisfy the conditions of (3.5), the state of stress and strain in the shell cannot be determined by the asymptotic method presented here.

The relations (3.5) according to which we establish a type of iterative process to be used in the given case, can also be expressed in terms of the characteristic quantities

$$\begin{aligned} &p^+ < q-2p \leftrightarrow hR \ll L^2, &&\text{(A)}: \ 1. \\ &\rho > q^+ + 2p^+ + 2p \leftrightarrow \mathscr{P} \ll E\frac{hL^2}{R^3}, &&2. \\ &\rho = q^+ + 2p^+ + 2p;\ p+p^+ > 0 \leftrightarrow \mathscr{P} \approx E\frac{hL^2}{R^3};\quad R \gg L; &&3. \\ &p^+ = q-2p \leftrightarrow hR \approx L^2, &&\text{(B)}: \ 4. \\ &p^+ > q-2p \leftrightarrow hR \gg L^2, &&5. \end{aligned} \tag{3.22}$$

$$\rho > 4(q-p) \leftrightarrow \mathscr{P} \ll E\left(\frac{h}{L}\right)^4, \qquad 6. \text{ (3.22 contd.)}$$

$$\rho = 4(q-p) \leftrightarrow \mathscr{P} \approx E\left(\frac{h}{L}\right)^4; \qquad 7.$$

$$\max\left[E\left(\frac{h}{L}\right)^4, E\left(\frac{hL^2}{R^3}\right)\right] \ll \mathscr{P} \lessapprox E\frac{h}{L}. \qquad \text{(C): } 8.$$

At the end of this section, it is convenient to recapitulate briefly the results of the foregoing investigation. The solution of the basic system of equations (1.13) subject to the surface conditions (1.18) is assumed to have the form of expansions (3.11), the coefficients of which can be determined from (3.13) and (3.16), provided $u_\alpha^{[s,0]}$ and $w^{[s,0]}$ are found from the main system of equations of the iterative process (3.17) [or (3.19) in case (B)]. The main system is linear in cases (A) and (B) if conditions 2 and 6 of (3.22) are satisfied, respectively. The main system of the zeroth order approximation consists of non-linear equations if condition 3 is case (A), 7 in case (B) and 8 in case (C) are fulfilled in (3.22).

4. ZEROTH ORDER APPROXIMATION

In this section we present the zeroth order approximation for the case (B) from (3.5) [by introduction of the additional symbol $\{i\}$, the resulting system of equations will be written in such a form that also cases (A) and (C) will be included]. The case (B) includes plates, shallow shells and shells loaded by a quickly varying surface load as it was shown in the preceding section. It is anticipated that the zeroth order approximation in this case will be in good agreement with the classical non-linear theory of plates and that of (quasi) shallow shells (as the shells loaded by quickly varying load, which are not included within the scope of classical non-linear theory, satisfy the assumptions of Koiter's theory of quasi-shallow shells [11], the formal comparison between the equations of the zeroth order approximation and those of [11] can be made). The relations (3.13) for $s = 0$ now take the form

$$1.\ \sigma^{\alpha 3}_{[0,n]} = -\frac{h}{2n}\nabla_\beta \sigma^{\alpha\beta}_{[0,n-1]} \qquad (n = 1,2);$$

$$2.\ \sigma^{33}_{[0,n]} = -\frac{h}{2n}[\{i\}\nabla_\alpha \sigma^{\alpha 3}_{[0,n-1]} + \{k\}b_{\alpha\beta}\sigma^{\alpha\beta}_{[0,n-1]} + \{l\}\sigma^{\alpha\beta}_{[0,n-1]}\nabla_\alpha\nabla_\beta w^{[0,0]}] \qquad (n = 1,2,3); \qquad (4.1)$$

$$3.\ \sigma^{\alpha\beta}_{[0,1]} = H^{\alpha\beta\gamma\varkappa}\nabla_\gamma u^{[0,1]}_\varkappa,$$

$$\sigma^{\alpha\beta}_{[0,0]} = H^{\alpha\beta\gamma\varkappa}(\nabla_\gamma u^{[0,0]}_\varkappa - \{k\}b_{\gamma\varkappa}w^{[0,0]} + \frac{1}{2}\{l\}\nabla_\gamma w^{[0,0]}\nabla_\varkappa w^{[0,0]});$$

$$4.\ u^{[0,1]}_\alpha = -\{i\}\frac{h}{2}\nabla_\alpha w^{[0,0]},$$

where in case (A)

$$\{i\} = 0, \quad \{k\} = 1,$$

$$\{l\} = 0 \quad \text{if} \quad \rho > q^+ + 2p^+ + 2p \qquad \left(\mathscr{P} \ll E\frac{hL^2}{R^3}\right), \tag{4.2}$$

$$\{l\} = 1 \quad \text{if} \quad \rho = q^+ + 2p^+ + 2p,\ p + p^+ > 0 \qquad \left(\mathscr{P} \approx E\frac{hL^2}{R^3}, \quad R \gg L\right),$$

in case (B)

$$\begin{aligned} &\{i\} = 1, \\ &\{k\} = 1 \quad \text{if} \quad p^+ = q - 2p \qquad (hR \approx L^2), \\ &\{k\} = 0 \quad \text{if} \quad p^+ > q - 2p \qquad (hR \gg L^2), \\ &\{l\} = 0 \quad \text{if} \quad \rho > 4(q-p) \qquad \left(\mathscr{P} \ll E\left(\frac{h}{L}\right)^4\right), \\ &\{l\} = 1 \quad \text{if} \quad \rho = 4(q-p) \qquad \left(\mathscr{P} \approx E\left(\frac{h}{L}\right)^4\right); \end{aligned} \tag{4.3}$$

and in case (C)

$$\{i\} = \{k\} = 0, \qquad \{l\} = 1. \tag{4.4}$$

As in the considered case we have

$$\lambda_{\alpha 3} = 2, \qquad \lambda_{33} = 3, \tag{4.5}$$

the equations of the main system (3.19) become

$$\begin{aligned} \sigma^{\alpha 3}_{[0,1]} &= \tfrac{1}{2}(P^{\alpha}_{[0]+} - P^{\alpha}_{[0]-}), \\ \sigma^{33}_{[0,3]} &= -\frac{1}{4}\left[P^3_{[0]+} - P^3_{[0]-} + \{i\}\frac{h}{2}\nabla_\alpha(P^{\alpha}_{[0]-} + P^{\alpha}_{[0]+})\right] - \frac{h}{4}R^{33}_{[0,1]}, \end{aligned} \tag{4.6}$$

where

$$R^{33}_{[0,1]} = \{k\} b_{\alpha\beta}\sigma^{\alpha\beta}_{[0,0]} + \{l\}\sigma^{\alpha\beta}_{[0,0]}\nabla_\alpha\nabla_\beta w^{[0,0]}. \tag{4.7}$$

To express (4.1) and (4.6) in a form suitable for the comparison with the equations of classical theory, we introduce the following notation

$$\sigma^{\alpha\beta}_{[0,0]} = \frac{1}{h}n^{\alpha\beta}_{[0]}, \qquad \sigma^{\alpha\beta}_{[0,1]} = \frac{6}{h^2}m^{\alpha\beta}_{[0]}, \qquad \sigma^{\alpha 3}_{[0,2]} = -\frac{3}{2h}q^{\alpha}_{[0]}. \tag{4.8}$$

Then equations (4.6) read

$$\nabla_\beta n^{\alpha\beta}_{[0]} = -(P^{\alpha}_{[0]+} - P^{\alpha}_{[0]-}), \tag{4.9}$$

$$\nabla_\alpha\nabla_\beta m^{\alpha\beta}_{[0]} + \{k\}b_{\alpha\beta}n^{\alpha\beta}_{[0]} + \{l\}n^{\alpha\beta}_{[0]}\nabla_\alpha\nabla_\beta w^{[0,0]} = -\left[P^3_{[0]+} - P^3_{[0]-} + \{i\}\frac{h}{2}\nabla_\alpha(P^{\alpha}_{[0]+} + P^{\alpha}_{[0]-})\right],$$

and to complete the system of the main equations we have to rewrite (4.1.3) (using (4.8))

$$n^{\alpha\beta}_{[0]} = hH^{\alpha\beta\gamma\varkappa}(\nabla_\gamma u^{[0,0]}_\varkappa - \{k\}b_{\gamma\varkappa}w^{[0,0]} + \tfrac{1}{2}\{l\}\nabla_\gamma w^{[0,0]}\,.\,\nabla_\varkappa w^{[0,0]}),$$
$$m^{\alpha\beta}_{[0]} = -\{i\}\frac{h^3}{12}H^{\alpha\beta\gamma\varkappa}\nabla_\gamma\nabla_\varkappa w^{[0,0]} \qquad (q^{\alpha}_{[0]} = \nabla_\beta m^{\alpha\beta}_{[0]}). \tag{4.10}$$

If we now substitute (4.10) into (4.9) we obtain three equations for three unknown quantities $u^{[0,0]}_\alpha$, $w^{[0,0]}$. In performing this manipulation we have to keep in mind that from (2.4) there follows

$$\nabla_\alpha(b_{\beta\gamma}F^{[s]}_{..}) = (\nabla_\alpha b_{\beta\gamma})F^{[s-p-p^+]}_{..} + b_{\beta\gamma}\nabla_\alpha F^{[s]}_{...}. \tag{4.11}$$

The resulting equations in the terms of the displacements now read

$$hH^{\alpha\beta\gamma\varkappa}[\nabla_\beta\nabla_\gamma u^{[0,0]}_\varkappa - \{k\}b_{\gamma\varkappa}\nabla_\beta w^{[0,0]} - \{m\}(\nabla_\beta b_{\gamma\varkappa})w^{[0,0]}$$
$$+\tfrac{1}{2}\nabla_\beta(\{l\}\nabla_\gamma w^{[0,0]}\,.\,\nabla_\varkappa w^{[0,0]})] = -P^{\alpha}_{[0]+} - P^{\alpha}_{[0]-}),$$
$$hH^{\alpha\beta\gamma\varkappa}\Big[\{i\}\frac{h^2}{12}\nabla_\alpha\nabla_\beta\nabla_\gamma\nabla_\varkappa w^{[0,0]} - (\{k\}b_{\alpha\beta} + \{l\}\nabla_\alpha\nabla_\beta w^{[0,0]})(\nabla_\gamma u^{[0,0]}_\varkappa \tag{4.12}$$
$$-\{k\}b_{\gamma\varkappa}w^{[0,0]} + \frac{1}{2}\{l\}\nabla_\gamma w^{[0,0]}\,.\,\nabla_\varkappa w^{[0,0]})\Big] = P^3_{[0]+} - P^3_{[0]-} + \{i\}\frac{h}{2}\nabla_\alpha(P^{\alpha}_{[0]-} + P^{\alpha}_{[0]+})$$

$(\{m\} = 1 \;\text{ if }\; p+p^+ = 0, \qquad \{m\} = 0 \;\text{ if }\; p+p^+ > 0).$

On account of (2.4) and of the relation

$$\nabla_\alpha\nabla_\beta F^{\gamma\varkappa} - \nabla_\beta\nabla_\alpha F^{\gamma\varkappa} = a^{\lambda\varkappa}(b_{\lambda\beta}b_{\alpha\varrho} - b_{\lambda\alpha}b_{\beta\varrho})F^{\gamma\varrho} + a^{\nu\gamma}(b_{\nu\beta}b_{\alpha\varrho} - b_{\nu\alpha}b_{\beta\varrho})F^{\varrho\varkappa}$$

we can write

$$\nabla_\alpha\nabla_\beta F^{\gamma\varkappa}_{[s]} = \nabla_\beta\nabla_\alpha F^{\gamma\varkappa}_{[s]} + M(a_{\alpha\beta}, b_{\alpha\beta})F^{\gamma\varkappa}_{[s-2p-2p^+]}, \tag{4.13}$$

where $M(a_{\alpha\beta}, b_{\alpha\beta})$ is a polynomial in arguments $a_{\alpha\beta}$, $b_{\alpha\beta}$ and their associated tensors (F may be an arbitrary tensor). We are now in position to put equations (4.9) and (4.10) in the usual form of the classical theory of shallow shells. The general solution of (4.9.1) may be given in terms of an AIRY stress function $\Phi_{[0]}$

$$n^{\alpha\beta}_{[0]} = N^{\alpha\beta}_{[0]} + \varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}\nabla_\lambda\nabla_\mu\Phi_{[0]} \qquad (\varepsilon^{11} = \varepsilon^{22} = 0,\, \varepsilon^{12} = -\varepsilon^{21} = a^{-\frac{1}{2}}), \tag{4.14}$$

where $N^{\alpha\beta}_{[0]}$ denote an arbitrary particular solution of (4.9.1).

After substituting (4.14) into (4.9.2) and after performing operations $\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}\nabla_\lambda\nabla_\mu$ in the relations

$$\frac{2}{h}F^{[0]}_{\alpha\beta\gamma\varkappa}n^{\gamma\varkappa}_{[0]} = \nabla_\alpha u^{[0,0]}_\beta + \nabla_\beta u^{[0,0]}_\alpha - 2\{k\}b_{\alpha\beta}w^{[0,0]} + \{l\}\nabla_\alpha w^{[0,0]}\nabla_\beta w^{[0,0]} \tag{4.15}$$

which are inverse to (4.10.1), we obtain two equations for the unknown quantities $\Phi_{[0]}$ and $w^{[0,0]}$ (if $p+p^+ > 0$):

$$\begin{aligned}
&\frac{1}{h}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}\varepsilon^{\gamma\nu}\varepsilon^{\varkappa\omega}F^{[0]}_{\alpha\beta\gamma\varkappa}\nabla_\lambda\nabla_\mu\nabla_\nu\nabla_\omega\Phi_{[0]}+\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}(\{k\}b_{\alpha\beta}\\
&\quad+\frac{1}{2}\{l\}\nabla_\alpha\nabla_\beta w^{[0,0]})\nabla_\lambda\nabla_\mu w^{[0,0]} = -\frac{1}{h}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}F^{[0]}_{\alpha\beta\gamma\varkappa}\nabla_\lambda\nabla_\mu N^{\gamma\varkappa}_{[0]};\\
&\quad-\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}(\{k\}b_{\alpha\beta}+\{l\}\nabla_\alpha\nabla_\beta w^{[0,0]})\nabla_\lambda\nabla_\mu\Phi_{[0]}+\{i\}\frac{h^3}{12}H^{\alpha\beta\gamma\varkappa}\nabla_\alpha\nabla_\beta\nabla_\gamma\nabla_\varkappa w^{[0,0]}\\
&\quad-\{l\}N^{\alpha\beta}_{[0]}\nabla_\alpha\nabla_\beta w^{[0,0]} = P^3_{[0]+}-P^3_{[0]-}+\{i\}\frac{h}{2}\nabla_\alpha(P^{\alpha}_{[0]-}+P^{\alpha}_{[0]+})+\{k\}b_{\alpha\beta}N^{\alpha\beta}_{[0]}.
\end{aligned} \tag{4.16}$$

Considering the case that

$$P^3 \approx P^\alpha \tag{4.17}$$

in (1.18), we have from (3.4)

$$P^{\alpha}_{[0]} \equiv 0, \tag{4.18}$$

and the resulting form of the system of main equations [(4.12) or (4.16)] is then simplified [in (4.16) $N^{\alpha\beta}_{[0]} \equiv 0$].

Finally, we remark that not only the right sides, but also the structure of the system of main equations for the higher order approximations differs from that of the zeroth order approximation. In the higher order approximations, the non-linear terms (e.g. of the type $\sigma^{\alpha\beta}_{[0,0]}\cdot w^{[0,0]}$) occurring in the zeroth order approximation are replaced by the linear terms (of the type $\sigma^{\alpha\beta}_{[s,0]}w^{[0,0]}+\sigma^{\alpha\beta}_{[0,0]}w^{[s,0]}$). From this it follows that the system of main equations for the higher order approximations is linear.

In addition we wish to note that by taking

$$r_{33}=0,\qquad r_{\alpha 3}=p+p^+,\qquad r_{\alpha\beta}=q^+,\qquad r_\alpha = 2q-3p-\rho,\qquad r_3 = 2q-2p-p^+-\rho,$$

instead of (3.6.A) we obtain the iterative process corresponding to the inextensional theory of shells (case (A) corresponds to the classical membrane theory of shells).

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Абстракт—Методом асимптотического интегрирования трехмерных (геометрически) нелинейных уравнений упругой среды построены двумерные итерационные процессы для определения составляющих тензора напряжения и вектора смещения в тонких анизотропных оболочках (пластинках). Условия, когда главная система уравнений итерационного процесса линейна или когда главная система нулевого приближения нелинейна, выражены через характеристические величины, которые характеризуют геометрические и материальные свойства оболочки (пластинки), интенсивность и изменяемость поверхностной нагрузки. Внимание уделяется только "внутренней" задаче—не рассматриваются краевые эффекты.